

Is There a Purely Fermionic Fractional Statistics?

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Abstract

Fermi statistics is formally extended to the case when energy levels are allowed to be fractionally occupied, which the Pauli principle does not categorically exclude. The *fractional* Fermi distribution obtained depends on the fractional occupation but otherwise has similar properties as the (integer) Fermi distribution. In the zero temperature limit both are identical.

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Fermi's ingenious truncation of the infinite sum in the partition function $Z_i = \sum_i \{\exp[(\mu - \epsilon_i)/k_B T]\}^{n_i}$ by boldly assuming binary occupations $n_i = [0, 1]$ only of energy states, ϵ_i with $i \in \mathbb{N}$, had profound consequences for the statistical mechanics of solids at low temperatures. It immediately led him to the proposal of his celebrated (Fermi) distribution $\langle n_i \rangle_F = \{1 + \exp[-(\mu - \epsilon_i)/k_B T]\}^{-1}$. His assumption was justified by the Pauli principle and for spin- $\frac{1}{2}$ particles was ultimately given its quantum mechanical interpretation based on the complete asymmetry of Fermionic wave functions. Discovery of the quantum Hall effect, in particular the fractional effect, had temporarily shaken Fermi statistics, leading to suggestion of anyon statistics, until Laughlin's (Laughlin, 1983) proposal of his wave function which

includes interaction with Bosonic fields.

What, on the other hand, happens, when sufficiently many states are available and the electrons would be allowed to fill such states only fractionally? As for an example one may think of gyrating electrons which bounce in a magnetic mirror geometry at frequency $\omega_b \ll \omega_{ce}$. In this case Landau levels $\epsilon_L = \omega_{ce}\hbar(L + \frac{1}{2})$, $L \in \mathbb{N}$, split into a number of bounce levels $\epsilon_b = \hbar\omega_b(b + \frac{1}{2})$, $b \in \mathbb{N}$, $b/L < \omega_{ce}/\omega_b$. The total electron energy $\epsilon_{b,L} = \epsilon_L + \epsilon_b$ in Landau level L is then shared by the two kinds of oscillatory states of the electron. Under these circumstances, all the electron energy is in the Landau levels at the mirror points, while in the minima of the magnetic field a substantial part of energy is transferred into the bounce levels. Theoretically, bounce levels might become only partially filled under these circumstances, even though only Fermionic states are involved, and fractional occupation of states may not necessarily mean that the Pauli principle is violated, when not involving Bosonic interactions. The occupation may still be a fraction below one which the Pauli principle not explicitly excludes. Though I am not aware of any observations of this case, in the following I rewrite the formalism for the fractional occupation case.

This is easily done, starting as usually (cf., e.g., Huang, 1987; Landau & Lifschitz, 1994) from the logarithm $\Omega_i[n_i]$ of the Γ -phase space volume corresponding to the occupation numbers $[n_i]$ of the particles in an ideal gas:

$$\Omega_i = -k_B T \log \sum_{n_i} \left(\exp \frac{\mu - \epsilon_i}{k_B T} \right)^{n_i}, \quad (1)$$

with μ chemical potential, $k_B T$ temperature, both in energy units, $\epsilon_i = p_i^2/2m$ particle energy, \mathbf{p} particle momentum, m mass, and the sum over Gibbs distributions in states n_i the canonical partition function Z_i .

Assume that the states can become fractionally occupied by Fermions alone. Occupation numbers $n_i > 1$ are excluded by the Pauli principle. Hence, fractional occupation implies that, given the interval $[0, \ell]$ with fixed natural number $\ell \in \mathbb{N}$, and j any integer such that $j \in [0, \ell]$, the thermodynamic potential Ω_i can be written

$$\Omega_i = -k_B T \log \sum_{j=0}^{\ell} \left(\exp \frac{\mu - \epsilon_i}{\ell k_B T} \right)^j, \quad j \in [0, \ell] \quad (2)$$

where $j = 0, \ell$ just reproduces the two Fermi occupations. In the intermediate interval the occupations follow the simple fractional chain $\{j/\ell\}$, with fixed $\ell \geq j$. Summation of the sum becomes simple matter since it represents a truncated geometric progression with ratio $\exp[(\mu - \epsilon_i)/\ell k_B T]$ yielding

$$\Omega_i = -k_B T \log \left\{ \frac{\exp(x_i \ell) - 1}{\exp(x_i) - 1} \right\}, \quad x_i \equiv \frac{\mu - \epsilon_i}{\ell k_B T}. \quad (3)$$

From here, taking the derivative $-\partial\Omega_i/\partial\mu$, the average *fractional* distribution $\langle n_i \rangle$ in the i th quantum state follows:

$$\langle n_i \rangle = \frac{\ell (e^{x_i} - 1) e^{x_i \ell} - e^{x_i} (e^{x_i \ell} - 1)}{\ell (e^{x_i} - 1) (e^{x_i \ell} - 1)}, \quad \ell \geq 1, \quad (4)$$

It is easily shown that the ordinary Fermi distribution $\langle n_i \rangle_F$ is reproduced for $\ell = 1$.

The fractional Fermi distribution Eq. (4) is a bit more complicated than the (integer) Fermi distribution. Apparently, it looks more like a Boson distribution. However, this is an illusion which becomes clear when checking its low and high temperature forms which agree with those for the Fermi distribution.

Thus the Fermionic property of the distribution is maintained. Obviously it results from the subtraction of two Bosonic distributions in Eq. (4). The

anti-symmetric property of the fractional many-particle Fermionic system being caused by subtraction.

From the fractional distribution all thermodynamic quantities like the equation of state can be derived formally defining the appropriate moments and introducing an appropriate new function

$$f_\ell(z) = \frac{4}{\sqrt{\pi}} \int_0^\infty y^2 dy \log \left(\frac{ze^{-y^2} - 1}{z^{1/\ell} e^{-y^2/\ell} - 1} \right), \quad \log z = \frac{\mu}{k_B T}. \quad (5)$$

which replaces the usual function $f_{3/2}(z)$.

One may even formally extend the fractional case to the extreme case of a continuity of fractional states. Then the sum in the expression for the thermodynamic potential Ω_i turns into an integral yielding that

$$\Omega_i = -k_B T \log \left\{ \frac{k_B T}{\mu - \epsilon_i} \left[\exp \left(\frac{\mu - \epsilon_i}{k_B T} \right) - 1 \right] \right\}. \quad (6)$$

and for the average distribution

$$\langle n_i \rangle = \frac{\mu - \epsilon_i}{k_B T} \left\{ \left[1 - \exp \left(-\frac{\mu - \epsilon_i}{k_B T} \right) \right]^{-1} - \frac{(k_B T)^2}{(\mu - \epsilon_i)^2} \right\}. \quad (7)$$

It is straightforward to show by expanding that, in the limit $k_B T \rightarrow 0$, it becomes

$$\langle n_i \rangle \simeq 1 - k_B T / (\mu - \epsilon_i), \quad \epsilon_i < \mu, \quad (8)$$

In summary, from the purely statistical mechanical point of view there is no obvious contradiction between the Pauli principle for Fermions and a fractional occupation of states. Statistical mechanics of Fermions, respecting the Pauli principle, does not categorically exclude the existence of fractionally occupied states as long as the fractional occupation number remains to be smaller than unity.

The fractional Fermi distribution Eq. (4) is a variant of the (integer occupation) Fermi distribution. The low and high temperature limits are the same, as is the definition of Fermi energy. Fermions occupying fractional states at $T = 0$ remain to be degenerate.

The present investigation so far lacks any application as well as its quantum mechanical justification. The assumed fractional occupation is solely Fermionic with no Bosons involved. This differs, for instance, from the Quantum Hall effect. The restriction to Fermions naturally implies a somewhat different physics, if fractional occupations should in some way be quantum mechanically realized. It, however, shows that statistical mechanics alone does not a priori inhibit purely Fermionic fractional occupations.

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